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Deterministic photon-photon $\sqrt{\text{SWAP}}$ gate using a Λ system

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We theoretically present a method to realize a deterministic photon-photon $\sqrt{\text{SWAP}}$ gate using a three-level Λ system interacting with single photons in reflection geometry. The Λ system is used completely passively as a temporary memory for a photonic qubit; the initial state of the Λ system may be arbitrary, and active control by auxiliary fields is unnecessary throughout the gate operations. These distinct merits make this entangling gate suitable for deterministic and scalable quantum computation.

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Single photons are promising candidates for implementing qubits in quantum computation due to their long coherence times. Furthermore, one-qubit gates such as Hadamard and NOT gates can be readily realized using linear-optical elements. Photonic qubits have the disadvantage that it is difficult to realize two-qubit controlled gates such as controlled-NOT gates due to the weak mutual interaction between photons [1]. This problem has been partially overcome by linear-optics quantum computation, which enables probabilistic controlled gates that successfully operate depending on the measurement results of ancillary photons [2,3].

In the quest for realizing deterministic controlled gates in quantum optics, a measurable nonlinear phase shift between single photons has been demonstrated using a cavity quantum electrodynamics system in the bad-cavity regime [4]. This system has the characteristic that radiation from the atom is nearly completely forwarded to a one-dimensional field that is determined by the radiation pattern of the cavity. Such one-dimensional configurations can be realized by a variety of physical systems, including a leaky resonator interacting with an atom or a quantum dot [5,6], a single emitter near a surface plasmon [7], and a superconducting qubit near a transmission line or a resonator [8–10]. Since the incident light inevitably interferes with the radiation from the system due to the reduced dimensionality, the effective light-matter interaction can be drastically enhanced under this configuration. Utilizing this property, several quantum devices have been proposed to date, such as controlled logic gates [11,12], quantum-state converters [13–15], and entanglers of photonic or material qubits [15–17]. These devices perform their tasks with the help of active quantum control of the material part (such as initialization [11–17], single-qubit rotation [11,12], and classical pumping [13–15]) and by measurements [16,17].

In the present study, we theoretically point out a unique potential of a three-level Λ system coupled to a one-dimensional photon field in the reflection geometry. (A Λ system is hereafter referred to as an "atom," although it can be implemented by other physical systems such as semiconductor quantum dots and superconducting Josephson junctions [18–20].) Use of the nonlinear photon-photon interaction has been regarded as a promising strategy for constructing a deterministic photon-photon gate, but high-fidelity operation is difficult by this approach due to the inherent phase noise [21]. Here we propose a scheme for a *deterministic* and *high-fidelity* photon-photon gate without using nonlinear interaction. We show that a photon-photon $\sqrt{\text{SWAP}}$ gate can be realized by using an atom completely *passively* as a temporary memory for photonic qubits; the initial state of the atom may be arbitrary including even mixed states, and active control of the atom is unnecessary throughout successive gate operations. These properties are quite advantageous when constructing scalable quantum networks. Furthermore, since the $\sqrt{\text{SWAP}}$ gate constitutes a universal set of quantum gates together with one-qubit gates [22], we hope that the proposed scheme will provide a vivid blueprint for future quantum computation that is deterministic and scalable.

The physical setup considered in this study is schematically illustrated in Fig. 1. The atom has two degenerate ground states ($|0\rangle$ and $|1\rangle$) and an excited state ($|2\rangle$), and the transition frequency is Ω. The $|1\rangle \leftrightarrow |2\rangle$ and $|0\rangle \leftrightarrow |2\rangle$ transitions in the atom are assisted respectively by horizontally (H) and vertically (V) polarized photons and the radiative decay rates for the $|2\rangle \to |1\rangle$ and $|2\rangle \to |0\rangle$ transitions are Γ_H and Γ_V. The total Hamiltonian including the atom and the photon field is given, under the rotating-wave approximation, by (putting $\hbar = c = 1$)

$$\mathcal{H} = \Omega\sigma_{22} + \int dk\left[kh_k^\dagger h_k + i\sqrt{\frac{\Gamma_H}{2\pi}}(\sigma_{21}h_k - h_k^\dagger\sigma_{12})\right] + \int dk\left[kv_k^\dagger v_k + i\sqrt{\frac{\Gamma_V}{2\pi}}(\sigma_{20}v_k - v_k^\dagger\sigma_{02})\right], \quad (1)$$

where $\sigma_{ij}(=|i\rangle\langle j|)$ is the atomic transition operator and h_k (v_k) is the annihilation operator for the H (V) polarized photon with wave number k. As shown in Fig. 1, we define the spatial coordinate r along the propagation direction of the photon and assign the negative (positive) region to the input (output) ports.

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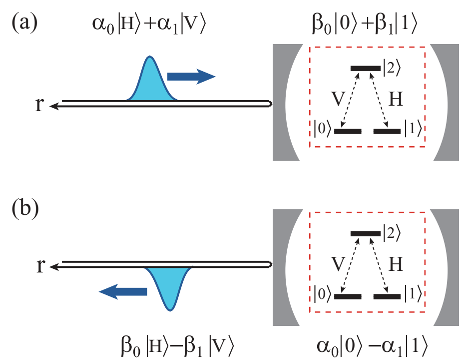

FIG. 1. (Color) Interaction between a Λ system and a single photon propagating in one dimension. (a) Initial state. The photonic and atomic qubits may be in arbitrary states. (b) Final state. The Λ system is completely deexcited through radiative decay. The photonic and atomic qubits can be completely swapped under appropriate conditions.

The real-space representation of the field operator $\widetilde{h}_r$ is defined as the Fourier transform of h_k by $\widetilde{h}_r = (2\pi)^{-1/2}\int dk e^{ikr} h_k$. Even when the atom is placed inside a cavity, we may base on Eq. (1) in the bad-cavity regime [4].

The initial states of the photon and the atom are given by $\alpha_0|H\rangle + \alpha_1|V\rangle$ and $\beta_0|0\rangle + \beta_1|1\rangle$, respectively [Fig. 1(a)]. We denote the wave packet of the input photon in the real-space representation by $f(r)$, which is normalized as $\int dr|f(r)|^2 = 1$. The four basis states of the input are then given by

$$|H,0\rangle = \int dr f(r)\widetilde{h}_r^\dagger|0\rangle, \tag{2}$$

$$|H,1\rangle = \int dr f(r)\widetilde{h}_r^\dagger|1\rangle, \tag{3}$$

$$|V,0\rangle = \int dr f(r)\widetilde{v}_r^\dagger|0\rangle, \tag{4}$$

$$|V,1\rangle = \int dr f(r)\widetilde{v}_r^\dagger|1\rangle. \tag{5}$$

The output states are determined by the Schrödinger equations, $|H,0\rangle \to e^{-i\mathcal{H}t}|H,0\rangle$, etc., where $\mathcal{H}$ is the Hamiltonian of Eq. (1) and the final time t is a sufficiently large time at which the atom is completely deexcited [Fig. 1(b)]. The time evolutions of $|H,0\rangle$ and $|V,1\rangle$ are trivial, since the input photon does not interact with the atom and therefore propagates freely. In contrast, the time evolutions of $|H,1\rangle$ and $|V,0\rangle$ are nontrivial, since the input photon may interact with the atom in these cases. The output state vectors are given by

$$|H,0\rangle \to \int dr g_1(r,t)\widetilde{h}_r^\dagger|0\rangle, \tag{6}$$

$$|H,1\rangle \to \int dr g_3(r,t)\widetilde{h}_r^\dagger|1\rangle - \int dr g_2(r,t)\widetilde{v}_r^\dagger|0\rangle, \tag{7}$$

$$|V,0\rangle \to \int dr g_4(r,t)\widetilde{v}_r^\dagger|0\rangle - \int dr g_2(r,t)\widetilde{h}_r^\dagger|1\rangle, \tag{8}$$

$$|V,1\rangle \to \int dr g_1(r,t)\widetilde{v}_r^\dagger|1\rangle, \tag{9}$$

where g_1, g_2, g_3, and g_4 are determined by [23]

$$g_1(r,t) = f(r-t), \tag{10}$$

$$g_2(r,t) = \sqrt{\Gamma_H\Gamma_V}\, s(t-r), \tag{11}$$

$$g_3(r,t) = f(r-t) - \Gamma_H s(t-r), \tag{12}$$

$$g_4(r,t) = f(r-t) - \Gamma_V s(t-r), \tag{13}$$

where $s(t)$ is the atomic coherence induced by the input photon, which evolves as

$$\frac{d}{dt}s(t) = \left(-i\Omega - \frac{\Gamma_H + \Gamma_V}{2}\right)s(t) + f(-t). \tag{14}$$

We first investigate a case in which the pulse length l of the input photon is sufficiently long to satisfy $l \gg \Gamma_{H,V}^{-1}$. In this case, Eq. (14) can be solved adiabatically. Denoting the detuning of the input photon by ω [namely, $f(r) \sim e^{i(\Omega+\omega)r}$], $s(t)$ is given by $s(t) = \frac{2}{\Gamma_H+\Gamma_V-2i\omega}f(-t)$. Substituting this into Eqs. (6)–(13) and neglecting the translational motion of the photon, the four basis states are transformed as follows on reflection:

$$|H,0\rangle \to |H,0\rangle, \tag{15}$$

$$|H,1\rangle \to \frac{\Gamma_V - \Gamma_H - 2i\omega}{\Gamma_H + \Gamma_V - 2i\omega}|H,1\rangle - \frac{2\sqrt{\Gamma_H\Gamma_V}}{\Gamma_H + \Gamma_V - 2i\omega}|V,0\rangle, \tag{16}$$

$$|V,0\rangle \to -\frac{2\sqrt{\Gamma_H\Gamma_V}}{\Gamma_H + \Gamma_V - 2i\omega}|H,1\rangle + \frac{\Gamma_H - \Gamma_V - 2i\omega}{\Gamma_H + \Gamma_V - 2i\omega}|V,0\rangle, \tag{17}$$

$$|V,1\rangle \to |V,1\rangle. \tag{18}$$

The case of $\Gamma_H = \Gamma_V$ is of particular interest as a quantum logic gate. When the input photon is in resonance with the atom ($\omega = 0$), this gate behaves as an atom-photon SWAP gate. As illustrated in Fig. 1, the quantum states of atomic and photonic qubits are exchanged on reflection as $(\alpha_0|H\rangle + \alpha_1|V\rangle)(\beta_0|0\rangle + \beta_1|1\rangle) \to (\beta_0|H\rangle - \beta_1|V\rangle)(\alpha_0|0\rangle - \alpha_1|1\rangle)$. Thus, the atom functions as a quantum memory for the photonic qubits. On the other hand, when the detuning of the input photon is set to the linewidth of the atom ($\omega = \pm\Gamma_H$), this gate behaves as an atom-photon $\sqrt{\text{SWAP}}$ gate. For example, when $\omega = -\Gamma_H$, $|H,1\rangle \to 2^{-1/2}(e^{i\pi/4}|H,1\rangle + e^{3i\pi/4}|V,0\rangle)$ and $|V,0\rangle \to 2^{-1/2}(e^{3i\pi/4}|H,1\rangle + e^{i\pi/4}|V,0\rangle)$, whereas $|H,0\rangle$ and $|V,1\rangle$ remain unchanged.

To observe the effects of a finite pulse length l, the shapes of g_1, g_2, and g_3 ($=g_4$) are plotted in Fig. 2 for the case of the atom-photon SWAP gate ($\Gamma_H = \Gamma_V$ and $\omega = 0$). The input mode function is assumed to be Gaussian, $f(r) = (2/\pi l^2)^{1/4}\exp(-r^2/l^2 + i\Omega r)$. It is observed that g_2 is slightly delayed relative to g_1 due to absorption and reemission by the atom. The delay time is of the order of Γ_H^{-1}. However, this delay becomes negligible when the input pulse is long ($l \gg \Gamma_H^{-1}$) as in Fig. 2(b). g_2 becomes almost identical to g_1, whereas g_3 vanishes. The average gate fidelities of the atom-photon SWAP and $\sqrt{\text{SWAP}}$ gates are respectively given by $\bar{F}_{\text{SWAP}} = [1 + |1 + \int dr g_2^* g_1|^2]/5$ and $\bar{F}_{\sqrt{\text{SWAP}}} = [1 + |1 + \frac{1+i}{4}\int dr(g_3^* + g_4^* - 2ig_2^*)g_1|^2]/5$ [24]. In Fig. 3, $\bar{F}_{\text{SWAP}}$ and $\bar{F}_{\sqrt{\text{SWAP}}}$ are plotted as functions of the pulse length l and the ratio Γ_V/Γ_H of the atomic decay rates. The conditions for achieving high-fidelity operations are given by $l \gg \Gamma_{H,V}^{-1}$ and $\Gamma_H/\Gamma_V \simeq 1$ for both gates.

These atom-photon gates can be turned into a deterministic photon-photon $\sqrt{\text{SWAP}}$ gate with a high fidelity. This implies

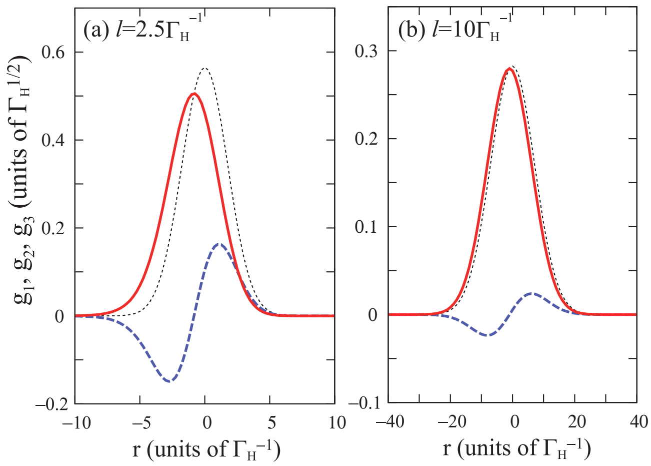

FIG. 2. (Color) Shapes of the output wave packets, g_1 (thin dotted line), g_2 (solid line), and g_3 (dashed line) for the case of the atom-photon SWAP gate ($\Gamma_H = \Gamma_V$ and $\omega = 0$). The natural phase factor $e^{i\Omega(r-t)}$ is removed. The pulse length is (a) $l = 2.5\Gamma_H^{-1}$ and (b) $l = 10\Gamma_H^{-1}$.

that deterministic all-optical quantum computation is possible, since a $\sqrt{\text{SWAP}}$ gate constitutes a universal set of quantum gates together with one-photon gates. Figure 4 shows a schematic illustration of the photon-photon $\sqrt{\text{SWAP}}$ gate. The initial state of the atom is arbitrary, and three photons (P_1, P_2, and P_3) are forwarded to the atom with sufficiently large time intervals between them. P_1 and P_3 are in resonance with the atom ($\omega = 0$), whereas P_2 is slightly detuned ($\omega = \pm\Gamma_H$). We assign P_1 and P_2 as the input qubits, and P_3 and P_2 as the output qubits. We can then confirm the following $\sqrt{\text{SWAP}}$ operation (for example, for $\omega = -\Gamma_H$ for P_2):

$$|H\rangle_1|H\rangle_2 \rightarrow |H\rangle_3|H\rangle_2, \quad (19)$$

$$|H\rangle_1|V\rangle_2 \rightarrow 2^{-1/2}(e^{i\pi/4}|H\rangle_3|V\rangle_2 + e^{-i\pi/4}|V\rangle_3|H\rangle_2), \quad (20)$$

$$|V\rangle_1|H\rangle_2 \rightarrow 2^{-1/2}(e^{-i\pi/4}|H\rangle_3|V\rangle_2 + e^{i\pi/4}|V\rangle_3|H\rangle_2), \quad (21)$$

$$|V\rangle_1|V\rangle_2 \rightarrow |V\rangle_3|V\rangle_2. \quad (22)$$

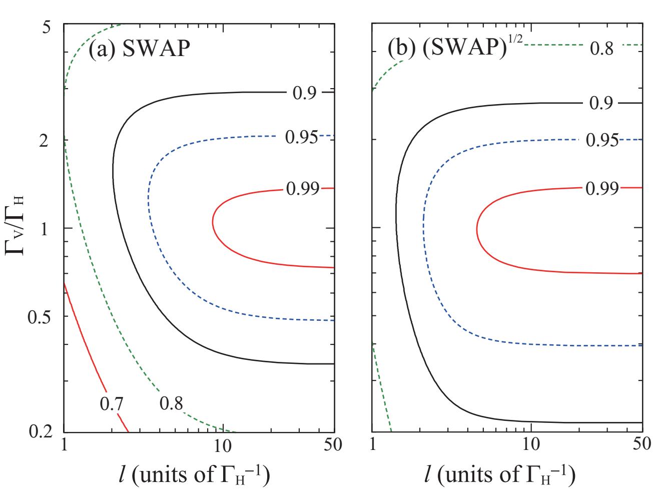

FIG. 3. (Color) Contour plots of the average gate fidelities for the atom-photon (a) SWAP ($\omega = 0$) and (b) $\sqrt{\text{SWAP}}$ ($\omega = -\Gamma_H$) gates, as functions of the pulse length l and Γ_V/Γ_H.

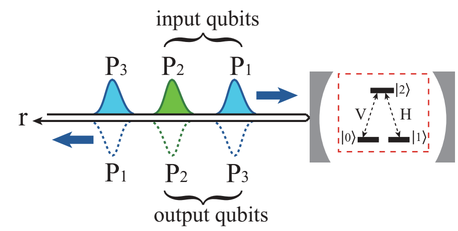

FIG. 4. (Color) Illustration of the photon-photon $\sqrt{\text{SWAP}}$ gate. P_1 and P_3 are in resonance with the atom ($\omega = 0$), whereas P_2 is detuned ($\omega = \pm\Gamma_H$). The input qubits are the polarization states of P_1 and P_2, whereas the output qubits are those of P_3 and P_2.

The initial states of the atom and P_3, both of which may be arbitrary, are respectively transferred to the final states of P_1 and the atom as

$$\alpha_0|0\rangle_a + \alpha_1|1\rangle_a \rightarrow \alpha_0|H\rangle_1 - \alpha_1|V\rangle_1, \quad (23)$$

$$\beta_0|H\rangle_3 + \beta_1|V\rangle_3 \rightarrow \beta_0|0\rangle_a - \beta_1|1\rangle_a, \quad (24)$$

where the subscript "a" denotes the atom. These states are unentangled with the output qubits and therefore do not affect the gate. They can also be recycled for subsequent gate operations.

We estimate the effects of practical noises and imperfections such as radiative loss, finite spin-coherence times, discrepancy between Γ_H and Γ_V, and finite pulse lengths, assuming that the Λ system is implemented by a charged quantum dot in a photonic crystal nanocavity [6]. The typical values of the cavity-QED parameters are $(g,\gamma,\kappa)\sim 2\pi \times (16,0.2,32)$ GHz, and therefore $\Gamma(\sim g^2/\kappa)\sim 2\pi \times 8$ GHz. Then, the photon loss rate is estimated at $\gamma/(\Gamma+\gamma) \sim 2.5\%$ per one gate operation. The gate fidelity can be estimated with a help of Fig. 3; when the pulse length l is 400 ps ($20\Gamma^{-1}$) and $\Gamma_H/\Gamma_V = 1.4$, for example, the fidelity of the photon-photon $\sqrt{\text{SWAP}}$ gate becomes $(0.99)^3 \sim 0.97$. The time intervals between photons should be shorter than the homogeneous spin-coherence time of the order of μs [25,26].

Five comments on this gate are in order. (i) This gate enables the $\sqrt{\text{SWAP}}$ operation between two photons having different frequencies. The SWAP operation between them can be achieved by using this gate twice. Therefore, the current scheme can be extended to construct the $\sqrt{\text{SWAP}}$ operation between two photons having *the same* frequency. (ii) Use of the nonlinear photon-photon interaction has been regarded as a promising strategy for constructing a deterministic photon-photon gate, but high-fidelity operation is difficult by this approach due to the inherent phase noise [21]. The present scheme does not rely on the optical nonlinearity and therefore enables high-fidelity operation, as demonstrated in Fig. 3. (iii) The present scheme does not use interference between photons. Therefore, high stability of optical paths, which is essential in many optical experiments, is not required. (iv) Initialization of the atom is unnecessary. Even if the atom is in a mixed state, it can be restored to a pure state automatically by the first input photon. Therefore, the quantum coherence of the atom should be maintained only during the interaction with three photons. (v) Throughout successive gate operations, there is no need for active control of the atom by auxiliary fields. Namely, the atom is used completely

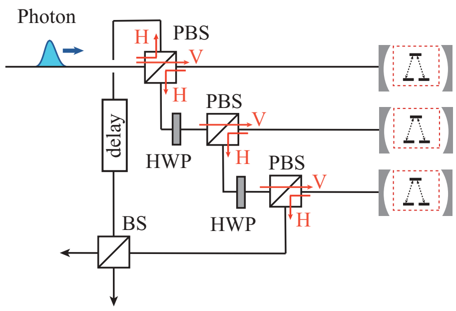

FIG. 5. (Color) Generation of a three-atom Greenberger-Horne-Zeilinger (GHZ) state. (PBS) polarization beam splitter; (HWP) half-wave plate; (BS) beam splitter. Initially, all atoms are in the $|0\rangle$ state. The polarization state of the input photon is $2^{-1/2}(|H\rangle + |V\rangle)$.

passively as a temporary quantum memory. These merits make the proposed scheme quite advantageous for constructing a scalable quantum network.

Finally, we observe in Fig. 5 that the atom-photon SWAP gates can be applied as a deterministic entangler of atomic qubits. Polarization beam splitters (PBS) transmit (reflect) V (H) polarized photons, and half-wave plates interchange the polarization of photons ($H \leftrightarrow V$). All atoms are initialized to be in $|0\rangle$ for the current purpose, and a single photon, $2^{-1/2}(|H\rangle + |V\rangle)$, is input in this circuit. The $|H\rangle$ component is then reflected by the first PBS and interacts with none of the atoms. In contrast, the $|V\rangle$ component interacts with all atoms inducing the $|0\rangle \rightarrow |1\rangle$ transitions. These two components are mixed by a beam splitter at the output to erase the which-path information. Then, the atoms form a three-qubit Greenberger-Horne-Zeilinger (GHZ) state, $2^{-1/2}(|0,0,0\rangle + |1,1,1\rangle)$. A remarkable feature is that entanglement is generated not by direct interaction between atoms but by mediation of quantum information by a photon. Therefore, the present scheme can be readily extended to generate many-qubit GHZ states. This is in contrast with conventional schemes, in which generating many-qubit entanglement becomes difficult as the number of qubits increases.

In summary, a deterministic photon-photon $\sqrt{\text{SWAP}}$ gate is proposed by using a Λ system as a temporary memory for photons. The proposed gate has the following distinct merits: the gate is free from the nonlinear photon-photon interaction and can operate with a high fidelity, and the Λ system is used completely passively and requires no active control throughout successive gate operations. The proposed gate is therefore suitable for constructing scalable quantum networks and computers.

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Supplementary material for "Deterministic photon-photon $\sqrt{\mathrm{SWAP}}$ gate using a lambda system"

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We present the mathematical details on the derivation of Eqs. (10)–(14) from Eqs. (1)–(5).

I. DERIVATION OF EQS. (10)–(14)

A. Heisenberg equations

From the Hamiltonian of Eq. (1), the Heisenberg equations for h_k and σ_{12} are given by

$$\frac{d}{d\tau}h_k = -ikh_k - \sqrt{\frac{\Gamma_H}{2\pi}}\sigma_{12}, \tag{A-1}$$

$$\frac{d}{d\tau}\sigma_{12} = -i\Omega\sigma_{12} + \sqrt{\Gamma_H}(\sigma_{11}-\sigma_{22})\widetilde{h}_0 + \sqrt{\Gamma_V}\sigma_{10}\widetilde{v}_0, \tag{A-2}$$

where $\widetilde{h}_0$ is the real-space representation of the field operator at $r=0$, namely, $\widetilde{h}_0 = (2\pi)^{-1/2}\int dk h_k$. From Eq. (A-1), $h_k(t)$ $(0<t)$ can be formally solved as

$$h_k(t) = h_k(0)e^{-ikt} - \sqrt{\frac{\Gamma_H}{2\pi}}\int_0^t d\tau\sigma_{12}(\tau)e^{-ik(t-\tau)}. \tag{A-3}$$

As the Fourier transform of the above equation, $\widetilde{h}_r(t)$ is given by

$$\widetilde{h}_r(t) = \widetilde{h}_{r-t}(0) - \sqrt{\Gamma_H}\theta(r)\theta(t-r)\sigma_{12}(t-r), \tag{A-4}$$

where $\theta(r)$ is a step function. Similarly, we have

$$\widetilde{v}_r(t) = \widetilde{v}_{r-t}(0) - \sqrt{\Gamma_V}\theta(r)\theta(t-r)\sigma_{02}(t-r). \tag{A-5}$$

These equations are known as the input-output relation [1]. In particular, the field operators at $r=0$ are given by

$$\widetilde{h}_0(t) = \widetilde{h}_{-t}(0) - \frac{\sqrt{\Gamma_H}}{2}\sigma_{12}(t). \tag{A-6}$$

$$\widetilde{v}_0(t) = \widetilde{v}_{-t}(0) - \frac{\sqrt{\Gamma_V}}{2}\sigma_{02}(t). \tag{A-7}$$

Substituting Eqs. (A-6) and (A-7) into Eq. (A-2) and using the symmetry of the system, we obtain

$$\frac{d}{d\tau}\sigma_{12} = \left(-i\Omega - \frac{\Gamma_H+\Gamma_V}{2}\right)\sigma_{12} + \sqrt{\Gamma_H}(\sigma_{11}-\sigma_{22})\widetilde{h}_{-\tau}(0) + \sqrt{\Gamma_V}\sigma_{10}\widetilde{v}_{-\tau}(0), \tag{A-8}$$

$$\frac{d}{d\tau}\sigma_{02} = \left(-i\Omega - \frac{\Gamma_H+\Gamma_V}{2}\right)\sigma_{02} + \sqrt{\Gamma_V}(\sigma_{00}-\sigma_{22})\widetilde{v}_{-\tau}(0) + \sqrt{\Gamma_H}\sigma_{01}\widetilde{h}_{-\tau}(0). \tag{A-9}$$

B. Temporal evolution of $|V,0\rangle$

We investigate the temporal evolution of the input states, Eqs. (2)–(5). As a nontrivial case, we consider here the evolution of $|V,0\rangle$. From Eqs. (4) and (8), the input and output state vectors are written as

$$|\varphi_{in}\rangle = \int dr f(r)\widetilde{v}_r^\dagger|0\rangle, \tag{A-10}$$

$$|\varphi_{out}\rangle = \int dr g_4(r;t)\widetilde{v}_r^\dagger|0\rangle - \int dr g_2(r;t)\widetilde{h}_r^\dagger|1\rangle. \tag{A-11}$$

These two state vectors are related by $|\varphi_{out}\rangle = e^{-i\mathcal{H}t}|\varphi_{in}\rangle$. The following two properties are useful in the arguments below: (I) $e^{-i\mathcal{H}t}|0\rangle = |0\rangle$ and $e^{-i\mathcal{H}t}|1\rangle = |1\rangle$ since $\mathcal{H}|0\rangle = \mathcal{H}|1\rangle = 0$. (II) $\widetilde{h}_r(0)|\varphi_{in}\rangle = 0$ and $\widetilde{v}_r(0)|\varphi_{in}\rangle = f(r)|0\rangle$, since the field commutators are given by $[\widetilde{v}_r, \widetilde{v}_{r'}^\dagger] = \delta(r-r')$ and $[\widetilde{h}_r, \widetilde{v}_{r'}^\dagger] = 0$.

From Eq. (A-11) and the property (I), g_2 and g_4 are determined by

$$g_2(r,t) = -\langle 1|\widetilde{h}_r|\varphi_{out}\rangle = -\langle 1|\widetilde{h}_r(t)|\varphi_{in}\rangle, \tag{A-12}$$
$$g_4(r,t) = \langle 0|\widetilde{v}_r|\varphi_{out}\rangle = \langle 0|\widetilde{v}_r(t)|\varphi_{in}\rangle, \tag{A-13}$$

where $A(t) = e^{i\mathcal{H}t}Ae^{-i\mathcal{H}t}$ (Heisenberg representation). Using Eqs. (A-4), (A-5), (A-10) and the property (II), g_2 and g_4 are recast into the following forms:

$$g_2(r,t) = \sqrt{\Gamma_H}\langle 1|\sigma_{12}(t-r)|\varphi_{in}\rangle, \tag{A-14}$$
$$g_4(r,t) = f(r-t) - \sqrt{\Gamma_V}\langle 0|\sigma_{02}(t-r)|\varphi_{in}\rangle. \tag{A-15}$$

$s_{02}(\tau) \equiv \langle 0|\sigma_{02}(\tau)|\varphi_{in}\rangle$ and $s_{12}(\tau) \equiv \langle 1|\sigma_{12}(\tau)|\varphi_{in}\rangle$ represent the atomic coherence induced by the input photon. Their equations of motion are given, from Eqs. (A-8)–(A-10) and the property (II), by

$$\frac{d}{d\tau}s_{02}(\tau) = \left(-i\Omega - \frac{\Gamma_H+\Gamma_V}{2}\right)s_{02}(\tau) + \sqrt{\Gamma_V}\langle 0|\sigma_{00}(\tau) - \sigma_{22}(\tau)|0\rangle f(-\tau), \tag{A-16}$$
$$\frac{d}{d\tau}s_{12}(\tau) = \left(-i\Omega - \frac{\Gamma_H+\Gamma_V}{2}\right)s_{12}(\tau) + \sqrt{\Gamma_V}\langle 1|\sigma_{10}(\tau)|0\rangle f(-\tau). \tag{A-17}$$

Since $\langle 0|\sigma_{00}(\tau) - \sigma_{22}(\tau)|0\rangle = \langle 1|\sigma_{10}(\tau)|0\rangle = 1$ from the property (I), $s_{02}(\tau)$ and $s_{12}(\tau)$ becomes identical. Introducing $s = s_{02}/\sqrt{\Gamma_V} = s_{12}/\sqrt{\Gamma_V}$, we obtain

$$g_2(r,t) = \sqrt{\Gamma_H\Gamma_V}s(t-r), \tag{A-18}$$
$$g_4(r,t) = f(r-t) - \Gamma_V s(t-r), \tag{A-19}$$
$$\frac{d}{d\tau}s(\tau) = \left(-i\Omega - \frac{\Gamma_H+\Gamma_V}{2}\right)s(\tau) + f(-\tau). \tag{A-20}$$

Thus, Eqs. (11), (13) and (14) of the main text are derived.

[1] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, New York, 1995), chapter 7.