# Semiclassical evaluation of the two-photon cross–Kerr effect

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When two distinguishable photons are simultaneously input into a nonlinear system, the output photons undergo nonlinear optical effects of the cross–Kerr type. Theoretical quantification of this two-photon cross–Kerr effect requires, in principle, a fully quantum-mechanical analysis involving heavy computation. In this paper, we propose a method for evaluating the two-photon cross–Kerr effect using a semiclassical optical response theory. The semiclassical method enables precise evaluation of the cross–Kerr effect with greatly reduced computation. The validity of the method is confirmed using a model nonlinear system.

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## I. INTRODUCTION

Control of light by light is one of the most challenging topics in current photonics technology. To date, all-optical switching of laser pulses has been demonstrated by several groups, by utilizing the optical nonlinearity inherent in optical fibers [1–6]. From the viewpoint of quantum information processing, extension of such technology to the single-photon domain, namely, control of a photon by a photon, is highly desirable in order to achieve an optical controlled phase gate, which is a key component for all-optical quantum computation [7,8]. However, an obstacle to all-optical control at the single-photon level is the inherent weakness of the two-photon nonlinear interaction. However, by utilizing the field-amplification effect of optical cavities, it was clearly demonstrated that considerable nonlinear effects can be obtained even when using weak input fields at the single-photon level [9,10]. Furthermore, owing to rapid progress in entangled-photon generation techniques [11–13], the availability of two-photon pulses has been greatly extended. Thus, quantification of nonlinear effects in two-photon states requires urgent theoretical investigation in order to determine the optimal design for two-photon nonlinear devices.

A straightforward way to evaluate the two-photon cross–Kerr effect is to analyze the quantum dynamics of two photons. Conventionally, the dynamics of several photons has been discussed theoretically, using effective Hamiltonians with single-mode approximations, in which it is implicitly assumed that there are no changes in the photonic pulse shapes. However, for precise evaluation of two-photon nonlinear effects, the changes in the pulse shapes must be taken into account, since considerable changes in the pulse shapes are inevitable when large nonlinear effects are obtained. In this case, the following two requirements must be fulfilled by the theory: (i) the photon field must be treated rigorously as a continuous field and (ii) both the optical media and the photon field must be treated quantum mechanically. However, such *fully* quantum-mechanical analyses of optical response require heavy numerical computation. Consequently, they have thus far been performed only for simple nonlinear optical media, such as two-level atoms [14–17]. In contrast, when classical light fields are used as the input, the optical responses can be analyzed within a semiclassical framework, in which the light fields can be treated as classical $c$-number fields. Owing to $c$-number treatment of light fields, semiclassical analyses are much simpler than fully quantum-mechanical analyses, and have therefore been applied to various types of realistic complicated systems [18,19].

Since the two-photon Fock state does not belong to classical fields, semiclassical results do not directly describe two-photon dynamics. However, considering that a classical field (coherent state) is composed of a superposition of Fock states, it is natural to expect that some information on two-photon dynamics is buried in the semiclassical results. From this perspective, it was revealed in Ref. [20] that two-photon nonlinearity can be evaluated by semiclassical methods, bypassing the need to perform fully quantum-mechanical calculations. The applicability of the theory presented in Ref. [20] was restricted to evaluation of the self–Kerr effect between two identical photons. However, the use of cross–Kerr-type nonlinearity between two distinguishable photons is also promising for the construction of two-photon devices, since such photons can be divided by optical devices, such as polarization beam splitters [21]. In this study, we present an extended theory for evaluating the cross–Kerr effect between two distinguishable photons within semiclassical optical response theory.

This study is organized as follows. In Sec. II, a measure of the cross–Kerr effect appearing in the output two-photon state is defined. In Sec. III, we derive a formula for evaluating the two-photon cross–Kerr effect by using semiclassical results. The validity of this formula is confirmed in Sec. IV, by calculating the cross–Kerr effect using two distinct formalisms, namely, fully quantum-mechanical and semiclassical ones, for a model cross–Kerr system. Section V summarizes the results.

## II. A MEASURE OF THE TWO-PHOTON CROSS–KERR EFFECT

The situation investigated in this study is illustrated in Fig. 1. Two photons with different polarizations (hereafter, $x$- and $y$-polarized photons) are inputted into a cross–Kerr me-

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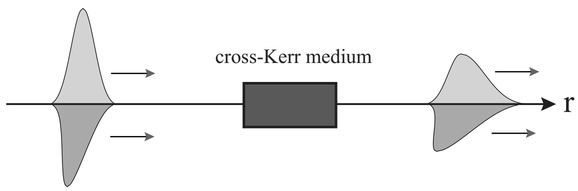


FIG. 1. Schematic illustration of the situation considered in this study. The cross–Kerr interaction between $x$- and $y$-polarized photons occurs in a cross–Kerr medium.

dium. Denoting the spatial wave functions of the $x$- and $y$-polarized photons by $f_a(r)$ and $f_b(r)$, respectively, the input state vector is given by

$$|\Psi_{\rm in}\rangle = \int dr_1 dr_2 f_a(r_1) f_b(r_2) a_{r_1}^\dagger b_{r_2}^\dagger |0\rangle, \qquad (1)$$

where $a_r^\dagger$ $(b_r^\dagger)$ represents a creation operator for an $x$-polarized ($y$-polarized) photon at $r$. The input wave functions, $f_a(r)$ and $f_b(r)$, are normalized as $\int dr|f_a(r)|^2 = \int dr|f_b(r)|^2 = 1$. After interaction with the cross–Kerr medium, the output state vector of these two photons is given by

$$|\Psi_{\rm out}\rangle = \int dr_1 dr_2 \tilde{g}_{ab}(r_1,r_2) a_{r_1}^\dagger b_{r_2}^\dagger |0\rangle. \qquad (2)$$

Note that, in contrast with the uncorrelated input state, the output wave function $\tilde{g}_{ab}(r_1,r_2)$ cannot be factored, in general, due to the nonlinear interaction between two photons. The output wave function $\tilde{g}_{ab}(r_1,r_2)$ is normalized as $\int dr_1 dr_2 |\tilde{g}_{ab}(r_1,r_2)|^2 = 1$.

In order to quantify the cross–Kerr effect appearing in the output state, the *linear* output state $|\Psi_{\rm out}^{\rm L}\rangle$ should be defined. Denoting the one-photon output wave functions by $\bar{f}_a(r)$ and $\bar{f}_b(r)$, which are the resultants of one-photon inputs $(\int dr f_a(r) a_r^\dagger|0\rangle$ and $\int dr f_b(r) b_r^\dagger|0\rangle)$, the *linear* output state vector is given by

$$|\Psi_{\rm out}^{\rm L}\rangle = \int dr_1 dr_2 \bar{f}_a(r_1) \bar{f}_b(r_2) a_{r_1}^\dagger b_{r_2}^\dagger |0\rangle. \qquad (3)$$

$\bar{f}_{a,b}(r)$ generally differs from $f_{a,b}(r)$ in shape due to linear dispersion, and $\bar{f}_{a,b}(r)$ is normalized as $\int dr|\bar{f}_a(r)|^2 = \int dr|\bar{f}_b(r)|^2 = 1$. As a measure of the cross–Kerr effect appearing in $|\Psi_{\rm out}\rangle$, we adopt the complex number $\alpha$, which is defined as the overlap between $|\Psi_{\rm out}\rangle$ and $|\Psi_{\rm out}^{\rm L}\rangle$

$$\alpha \equiv \langle \Psi_{\rm out}^{\rm L}|\Psi_{\rm out}\rangle = \int dr_1 dr_2 \bar{f}_a^*(r_1)\bar{f}_b^*(r_2)\tilde{g}_{ab}(r_1,r_2). \qquad (4)$$

By definition, $|\alpha| \le 1$. When $|\alpha| = 1$ $(\alpha = e^{i\theta})$, it necessarily follows that $|\Psi_{\rm out}\rangle = e^{i\theta}|\Psi_{\rm out}^{\rm L}\rangle$. In this case, the phase $\theta$ can be interpreted as the nonlinear phase shift, which is applicable to optical phase gates. However, $|\alpha|$ is not necessarily unity [16,17], since the spatial envelope of the two-photon output wave function $\tilde{g}_{ab}(r_1,r_2)$ generally differs from that of the linear one, $\bar{f}_a(r_1)\bar{f}_b(r_2)$.

## III. SEMICLASSICAL EVALUATION OF TWO-PHOTON CROSS–KERR EFFECT

### A. Optical response to classical fields

In this section, we consider the optical response to the classical input fields. When the amplitude of the input field is given by $f_a(r)$ for the $x$ polarization and by $f_b(r)$ for the $y$ polarization, the state vector describing this classical field is given by

$$|\Phi_{\rm in}\rangle = \mathcal{N}\exp\left(\int dr f_a(r) a_r^\dagger\right)\exp\left(\int dr f_b(r) b_r^\dagger\right)|0\rangle, \qquad (5)$$

where $\mathcal{N}$ is the normalization constant defined by $\mathcal{N} = \exp[-2^{-1}\int dr(|f_a(r)|^2 + |f_b(r)|^2)]$. For up to two-photon components, this state can be expanded as follows:

$$|\Phi_{\rm in}\rangle = \mathcal{N}(|0\rangle + |A_{\rm in}\rangle + |B_{\rm in}\rangle + 2^{-1/2}|AA_{\rm in}\rangle + 2^{-1/2}|BB_{\rm in}\rangle + |AB_{\rm in}\rangle), \qquad (6)$$

where

$$|A_{\rm in}\rangle = \int dr f_a(r) a_r^\dagger|0\rangle, \qquad (7)$$

$$|B_{\rm in}\rangle = \int dr f_b(r) b_r^\dagger|0\rangle, \qquad (8)$$

$$|AA_{\rm in}\rangle = 2^{-1/2}\int dr_1 dr_2 f_a(r_1) f_a(r_2) a_{r_1}^\dagger a_{r_2}^\dagger|0\rangle, \qquad (9)$$

$$|BB_{\rm in}\rangle = 2^{-1/2}\int dr_1 dr_2 f_b(r_1) f_b(r_2) b_{r_1}^\dagger b_{r_2}^\dagger|0\rangle, \qquad (10)$$

$$|AB_{\rm in}\rangle = \int dr_1 dr_2 f_a(r_1) f_b(r_2) a_{r_1}^\dagger b_{r_2}^\dagger|0\rangle. \qquad (11)$$

The factors $2^{-1/2}$ on the right-hand sides of Eqs. (9) and (10) are the normalization constants. After interaction with the medium, the above five state vectors are transformed as follows:

$$|A_{\rm in}\rangle \to |A_{\rm out}\rangle = \int dr \bar{f}_a(r) a_r^\dagger|0\rangle, \qquad (12)$$

$$|B_{\rm in}\rangle \to |B_{\rm out}\rangle = \int dr \bar{f}_b(r) b_r^\dagger|0\rangle, \qquad (13)$$

$$|AA_{\rm in}\rangle \to |AA_{\rm out}\rangle = 2^{-1/2}\int dr_1 dr_2 \tilde{g}_{aa}(r_1,r_2) a_{r_1}^\dagger a_{r_2}^\dagger|0\rangle, \qquad (14)$$

$$|BB_{\rm in}\rangle \to |BB_{\rm out}\rangle = 2^{-1/2}\int dr_1 dr_2 \tilde{g}_{bb}(r_1,r_2) b_{r_1}^\dagger b_{r_2}^\dagger|0\rangle, \qquad (15)$$

$$|AB_{\mathrm{in}}\rangle \rightarrow |AB_{\mathrm{out}}\rangle = \int dr_1 dr_2 \tilde{g}_{ab}(r_1,r_2) a_{r_1}^{\dagger} b_{r_2}^{\dagger}|0\rangle, \quad (16)$$

where $\tilde{g}_{aa}$ and $\tilde{g}_{bb}$ denote the output two-photon wave functions when two identical photons having the same polarization are input. Since both self– and cross–Kerr effects are usually inherent in nonlinear media, $\tilde{g}_{aa} \neq \bar{f}_a \bar{f}_a$ and $\tilde{g}_{bb} \neq \bar{f}_b \bar{f}_b$, in general, due to the self–Kerr effect [20]. As a result of the linearity in the quantum time evolution (Schrödinger equation), the output state vector is given by

$$|\Phi_{\mathrm{out}}\rangle = \mathcal{N}(|0\rangle + |A_{\mathrm{out}}\rangle + |B_{\mathrm{out}}\rangle + 2^{-1/2}|AA_{\mathrm{out}}\rangle + 2^{-1/2}|BB_{\mathrm{out}}\rangle + |AB_{\mathrm{out}}\rangle). \quad (17)$$

The amplitude of the output field can readily be calculated from the output state vector of Eq. (17): the amplitudes of the $x$- and $y$-polarized components are given by $\langle a_r\rangle = \langle \Phi_{\mathrm{out}}|a_r|\Phi_{\mathrm{out}}\rangle$ and $\langle b_r\rangle = \langle \Phi_{\mathrm{out}}|b_r|\Phi_{\mathrm{out}}\rangle$, respectively. Hereafter, we are concerned only with $\langle a_r\rangle$ and denote it by $F_{xy}(r)$. (The subscript $xy$ indicates that the input field has both $x$- and $y$-polarized components.) The linear and third-order components of $F_{xy}(r)$ are given by

$$F_{xy}^{(1)}(r) = \bar{f}_a(r), \quad (18)$$

$$F_{xy}^{(3)}(r) = \int dr' \bar{f}_a^{*}(r')\tilde{g}_{aa}(r,r') + \int dr' \bar{f}_b^{*}(r')\tilde{g}_{ab}(r,r') - \bar{f}_a(r)\int dr'[|f_a(r')|^2 + |f_b(r')|^2], \quad (19)$$

From Eq. (19), it can be seen that the third-order output field $F_{xy}^{(3)}$ contains contracted information on the two-photon output wave functions, $\tilde{g}_{aa}$ and $\tilde{g}_{ab}$.

For later use, we also consider a case in which only an $x$-polarized classical field is inputted. The output amplitude in this case, which is denoted by $F_x(r)$, is simply given by putting $f_b(r) \rightarrow 0$ in Eqs. (18) and (19). Namely,

$$F_{x}^{(1)}(r) = \bar{f}_a(r), \quad (20)$$

$$F_{x}^{(3)}(r) = \int dr' \bar{f}_a^{*}(r')\tilde{g}_{aa}(r,r') - \bar{f}_a(r)\int dr'|f_a(r')|^2. \quad (21)$$

It should be remarked again that these output amplitudes ($F_{xy}^{(1)}$, $F_{xy}^{(3)}$, $F_{x}^{(1)}$ and $F_{x}^{(3)}$) can be calculated by semiclassical optical response theory.

### B. Two-photon cross–Kerr effect

Our objective is to evaluate the measure $\alpha$ of the two-photon cross–Kerr effect defined in Eq. (4), from quantities obtainable using semiclassical theory, namely, the amplitudes of the output field. It can be readily confirmed that the quantity $\alpha'$ given by

$$\alpha' = 1 + \int dr[F_{xy}^{(1)}(r)]^{*}F_{xy}^{(3)}(r) - \int dr[F_{x}^{(1)}(r)]^{*}F_{x}^{(3)}(r) \quad (22)$$

is identical to $\alpha$ of Eq. (4). Thus, by calculating $\alpha'$ using semiclassical theory, the two-photon cross–Kerr effect can be evaluated without the need to do fully quantum-mechanical calculations.

There are three remarks on Eq. (22): (i) When both $x$- and $y$-polarized fields are input, both the self– and cross–Kerr effects are reflected in the nonlinear output field, $F_{xy}^{(3)}$. More specifically, the self–Kerr effect appears in the first term on the right-hand side of Eq. (19), while the cross–Kerr effect appears in the second term. In order to extract the cross–Kerr effect, the self–Kerr effect is subtracted, as represented by the third term on the right-hand side of Eq. (22). (ii) Equation (22) is constructed by only the $x$-polarized amplitudes of the output field. However, by interchanging the roles of $x$- and $y$-polarizations, semiclassical evaluation of the two-photon cross–Kerr effect is also possible through the $y$-polarized amplitudes. (iii) In deriving the above formula, except for the assumption that the output photon state remains as a pure state [see Eqs. (12)–(16)], no system-dependent features are required. Equation (22) is therefore widely applicable to various cross–Kerr media in the dissipation-free limit.

## IV. CONFIRMATION

This section is devoted to verifying the validity of the proposed formula. To this end, using a specific model of a cross–Kerr system, we will evaluate the two-photon cross–Kerr effect by two different methods. In one method, the linear and nonlinear two-photon output states ($|\Psi_{\mathrm{out}}^{\mathrm{L}}\rangle$ and $|\Psi_{\mathrm{out}}\rangle$) are calculated in a fully quantum-mechanical fashion, and the two-photon cross–Kerr effect is directly evaluated using its definition given in Eq. (4). In the other method, after calculating the output fields against classical input fields, the cross–Kerr effect is evaluated using Eq. (22).

### A. Interacting boson model

As the simplest cross–Kerr system, we employ two interacting bosonic particles having the same transition energy $\Omega$. Putting $\hbar = c = 1$ and choosing $\Omega$ as the origin of energy, the Hamiltonian of the system, including the photon field, is given by

$$\mathcal{H} = \lambda a^{\dagger}ab^{\dagger}b + \int dk\left[k a_k^{\dagger}a_k + \sqrt{\frac{\Gamma}{2\pi}}(a^{\dagger}a_k + \mathrm{H.c.})\right] + \int dk\left[k b_k^{\dagger}b_k + \sqrt{\frac{\Gamma}{2\pi}}(b^{\dagger}b_k + \mathrm{H.c.})\right], \quad (23)$$

where $a$ and $b$ are the annihilation operators of bosons, $a_k$ ($b_k$) denotes the annihilation operator for the $x$- ($y$-) polarized photon with wave number $k$, and $\lambda$ represents the coupling constant between two bosons. The real-space annihilation operator $a_r$, which appeared in previous sections [in Eq. (1), etc.], is given by $a_r = (2\pi)^{-1/2}\int dk e^{ikr}a_k$.

There are two comments regarding this system. (i) The boson $a$ ($b$) is coupled only to the $x$- ($y$-) polarized field. The cross–Kerr interaction between $x$- and $y$-polarized photons originates in the interaction between the oscillators, $\lambda a^\dagger a b^\dagger b$. Note that the self–Kerr interaction between identical photons is absent in the present system. (ii) In the Hamiltonian of Eq. (23), optical media (bosons) are directly coupled to one-dimensional photon fields by the coupling constant $\Gamma$. Such direct coupling can be attained in the weak coupling regime of cavity-QED systems. In terms of conventional cavity-QED parameters $g$ and $\kappa$, $\Gamma$ is given by $\Gamma = 4g^2/\kappa$ [9].

### B. Fully quantum-mechanical evaluation

The procedure for the fully quantum-mechanical evaluation is outlined as follows. The photon field is discretized by imposing a periodic boundary condition on $r$. If the period is large enough, then imposition of the periodic boundary condition has no influence on the numerical results. In addition, the number of photonic modes is made finite by introducing a cutoff wave number $k_{\mathrm{cut}}$ and treating only photonic modes satisfying $|k| < k_{\mathrm{cut}}$. After this finitization of the Hamiltonian, the output state $|\Psi_{\mathrm{out}}\rangle$ is calculated by solving the Schrödinger equation in the two-quanta Hilbert space. Linear output $|\Psi_{\mathrm{out}}^{\mathrm{L}}\rangle$ is obtained by putting $\lambda = 0$ in the Hamiltonian. From these two states, the parameter $\alpha$ is evaluated using Eq. (4).

### C. Semiclassical evaluation

In order to apply the semiclassical evaluation formula, we here analyze the linear and nonlinear responses when classical light pulses are input. The initial state vector is given by Eq. (5). The Heisenberg equation for $a$ and $b$ are given, with the help of the input-output formalism [22–25], by

$$\dot{a} = -\left(\frac{\Gamma}{2}\right)a - i\lambda b^\dagger a b - i\sqrt{\Gamma} a_{t_0 - t}(t_0), \tag{24}$$

$$\dot{b} = -\left(\frac{\Gamma}{2}\right)b - i\lambda a^\dagger a b - i\sqrt{\Gamma} b_{t_0 - t}(t_0), \tag{25}$$

where $t_0$ denotes the initial moment, and $a_{t_0-t}(t_0)$ $[b_{t_0-t}(t_0)]$ is the initial annihilation operator for the $x$- [$y$-] polarized photon field at $r = t_0 - t$. The photon-field operators at time $t$ are given by

$$a_r(t) = a_{r-t+t_0}(t_0) - i\sqrt{\Gamma} a(t-r), \tag{26}$$

$$b_r(t) = b_{r-t+t_0}(t_0) - i\sqrt{\Gamma} b(t-r). \tag{27}$$

By taking the expectation values with respect to the initial state vector $|\Phi_{\mathrm{in}}\rangle$ given by Eq. (5), the operator equations of Eqs. (24) and (25) are transformed into $c$-number equations of motion for $\langle a\rangle$ and $\langle b\rangle$. It should be remarked that the initial photon operator $a_r(t_0)$ is replaceable with a $c$-number $f_a(r)$, since the initial state is a coherent state and, therefore, $a_r(t_0)|\Phi_{\mathrm{in}}\rangle = f_a(r)|\Phi_{\mathrm{in}}\rangle$. Thus, familiar semiclassical equations of motion are obtained. Denoting the linear components of $\langle a\rangle$ and $\langle b\rangle$ by $\langle a^{(1)}\rangle$ and $\langle b^{(1)}\rangle$, they evolve as

$$\frac{d}{dt}\langle a^{(1)}\rangle = -\left(\frac{\Gamma}{2}\right)\langle a^{(1)}\rangle - i\sqrt{\Gamma} f_a(t_0 - t), \tag{28}$$

$$\frac{d}{dt}\langle b^{(1)}\rangle = -\left(\frac{\Gamma}{2}\right)\langle b^{(1)}\rangle - i\sqrt{\Gamma} f_b(t_0 - t). \tag{29}$$

For the determination of the response up to the third order, three second-order quantities ($\langle ab\rangle$, $\langle b^\dagger b\rangle$, and $\langle b^\dagger a\rangle$) and two third-order quantities ($\langle a^{(3)}\rangle$ and $\langle b^\dagger a b\rangle$) are relevant. The equations of motion for these quantities are given by

$$\frac{d}{dt}\langle ab\rangle = -(\Gamma + i\lambda)\langle ab\rangle - i\sqrt{\Gamma}[f_b(t_0 - t)\langle a^{(1)}\rangle + f_a(t_0 - t)\langle b^{(1)}\rangle], \tag{30}$$

$$\frac{d}{dt}\langle b^\dagger b\rangle = -\Gamma\langle b^\dagger b\rangle + i\sqrt{\Gamma}[f_b^*(t_0 - t)\langle b^{(1)}\rangle - \mathrm{c.c.}], \tag{31}$$

$$\frac{d}{dt}\langle b^\dagger a\rangle = -\Gamma\langle b^\dagger a\rangle + i\sqrt{\Gamma}[f_b^*(t_0 - t)\langle a^{(1)}\rangle - f_a(t_0 - t)\langle b^{(1)}\rangle^*], \tag{32}$$

$$\frac{d}{dt}\langle a^{(3)}\rangle = -\left(\frac{\Gamma}{2}\right)\langle a^{(3)}\rangle - i\lambda\langle b^\dagger a b\rangle, \tag{33}$$

$$\frac{d}{dt}\langle b^\dagger a b\rangle = -\left(\frac{3\Gamma}{2} + i\lambda\right)\langle b^\dagger a b\rangle + i\sqrt{\Gamma}[f_b^*(t_0 - t)\langle ab\rangle - f_a(t_0 - t)\langle b^\dagger b\rangle - f_b(t_0 - t)\langle b^\dagger a\rangle]. \tag{34}$$

Finally, the output field amplitudes are given by

$$F_{xy}^{(1)}(r) = \langle a_r^{(1)}(t)\rangle = f_a(r - t + t_0) - i\sqrt{\Gamma}\langle a^{(1)}(t - r)\rangle, \tag{35}$$

$$F_{xy}^{(3)}(r) = \langle a_r^{(3)}(t)\rangle = -i\sqrt{\Gamma}\langle a^{(3)}(t - r)\rangle, \tag{36}$$

where the final time $t$ should be chosen as an arbitrary time sufficiently after the interaction. It is observed that the above output amplitudes are obtained by solving the simultaneous equations of motion for seven quantities, which is obviously a much reduced task compared to that for the fully quantum-mechanical method outlined in Sec. IV B.

In order to use Eq. (22), we must also determine $F_x^{(1)}$ and $F_x^{(3)}$, that is, the output amplitudes when only an $x$-polarized pulse is input. They are obtained by putting $f_b \to 0$ in Eqs. (28)–(34). Then, it is readily confirmed that the nonlinear response $F_x^{(3)}$ vanishes as expected and that the subtraction of the self–Kerr effect is not required in the present model. However, this is specific to the interacting boson model of Eq. (23) and the subtraction procedure is necessary in general.

### D. Results

Here, we compare the numerical results obtained by the two distinct formalisms. Regarding the pulse shapes of input

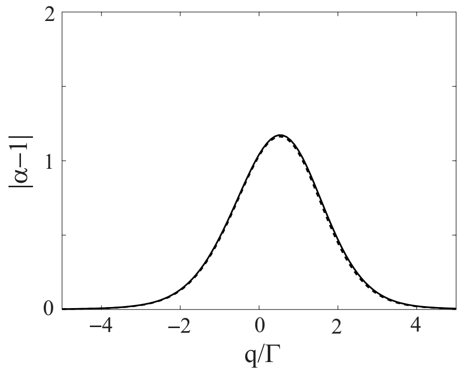


FIG. 2. Dependence of $|\alpha-1|$ on the energy $q$ of input photons. The pulse length $d$ and the coupling $\lambda$ between bosons are chosen as $d=\Gamma^{-1}$ and $\lambda=3\Gamma$. The dotted line shows the results obtained by the semiclassical method. The solid line shows the results obtained by fully quantum-mechanical calculations, where the cutoff energy is chosen as $k_c=16.3\Gamma$.

photons, we consider the case in which the $x$- and $y$-polarized photons have the same Gaussian forms given by

$$f_a(r)=f_b(r)=\left(\frac{2}{\pi d^2}\right)^{1/4}\exp\left[-\left(\frac{r}{d}\right)^2+iqr\right], \qquad (37)$$

which is characterized by the coherence length $d$ and the central frequency $q$. The two-photon cross–Kerr effect is expected to be maximized when $d\sim\Gamma^{-1}$ and $q\sim 0$, since under these conditions the input photons are most effectively absorbed by the oscillators [16].

In Fig. 2, as a real quantity representing the strength of the two-photon cross–Kerr effect, $|\alpha-1|$ is plotted as a function of the input photon frequency $q$, by fixing the pulse length at $d=\Gamma^{-1}$. As expected, the cross–Kerr effect is strong around the resonance frequency of bosons, $q\sim 0$. However, since $\lambda$ is positive in Fig. 2, the energies of the oscillators are blueshifted and $|\alpha-1|$ is maximized by photons with $q>0$.

It is apparent that the results of the semiclassical (dotted line) and quantum-mechanical (solid line) methods are in good agreement. However, a slight discrepancy is observed between the two results. This discrepancy should be attributed to the imperfect nature of the quantum-mechanical method due to the introduction of the cutoff wave number $k_{\mathrm{cut}}$, since it was confirmed numerically that the discrepancy decreases as $k_{\mathrm{cut}}$ is increased. Thus, if the quantum-mechanical calculation could be carried out rigorously ($k_{\mathrm{cut}}\to\infty$), then these two results are expected to show complete agreement. These observations demonstrate the validity of the semiclassical evaluation method, as well as its potential as a simple and accurate tool for the evaluation of the two-photon cross–Kerr effect.

## V. SUMMARY

When $x$- and $y$-polarized photons are input into a nonlinear medium simultaneously, the cross–Kerr effect appears in the output photons (see Fig. 1). The measure of the cross–Kerr effect is defined in Sec. II, as the overlap between the linear and nonlinear output wave functions, as given by Eq. (4). In Sec. III, we have proposed a method for evaluating the two-photon cross-Kerr effect using semiclassical optical response theory, bypassing the need to perform fully quantum-mechanical calculations. The prescription is as follows: calculate the linear and the third-order nonlinear components of the output field, for the case in which the input classical light pulse has both $x$ and $y$ components, and also for the case in which the input pulse has only an $x$ component; then, evaluate the two-photon cross–Kerr effect using Eq. (22). In Sec. IV, taking the interacting boson model as an example of a cross–Kerr system, the semiclassical and quantum-mechanical results are compared (see Fig. 2) and the validity of the semiclassical method is demonstrated.

The merits of the proposed method are summarized in two points. One merit is that, as observed in Sec. IV, the method enables accurate evaluation of the two-photon cross–Kerr effect with reduced calculations. The other merit is that the method is applicable to a wide range of nonlinear systems, since no system-dependent features are assumed in the derivation presented in Sec. III. Thus, the semiclassical evaluation method could serve as a convenient theoretical tool in future photonics technology in the quantum domain.

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