Relation between conventional and dynamical formalisms in the quantum Zeno effect

Kazuki Koshino*
CREST, Japan Science and Technology Agency, 4-1-8 Honcho, Kawaguchi, Saitama 332-0012, Japan
and Department of Materials Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan
(Received 5 January 2005; published 25 March 2005)

The measurement-modified decay rate is calculated in two distinct formalisms, i.e., the conventional formalism and the dynamical formalism. The relation between the two formalisms is clarified, by recasting the decay rates obtained by the two formalisms into a unified form. It is shown that the dynamical formalism reproduces the conventional results only under the condition that the apparatus detects the decayed states with an identical response time.

DOI: 10.1103/PhysRevA.71.034104 PACS number(s): 03.65.Xp, 06.20.Dk, 03.65.Yz

I. INTRODUCTION

It was theoretically predicted that frequent measurements on an unstable quantum state would suppress the decay of that state, which is known as the quantum Zeno effect (QZE) [1]. Later, it was pointed out that the opposite effect—acceleration of decay—may sometimes be caused by frequent measurements, which is known as the anti-Zeno effect (AZE) [2–5]. The experimental observations of the QZE were restricted to oscillating quantum systems in the early days [6], but, recently, both the QZE and AZE were successfully observed using an irreversibly decaying system [7,8]. Besides academic interest, practical applications of repeated measurements are also proposed [9–11], which make this research field more attractive.

In the theoretical analysis of the QZE and AZE, there are two distinct formalisms. In one formalism, measurements are simply described by the projection postulate, assuming that instantaneous and ideal measurements are repeatedly performed on the target system. Thus, the dynamics of the target system is calculated by combining the unitary dynamics of the system and the projective operations at every instant of measurement. Originally, the QZE was predicted based on this formalism [1], which is called the *conventional* formalism in this study. In the other formalism, in order to discuss the effects of measurements, one explicitly considers the interaction between the target quantum system and the measurement apparatus, and examines the unitary dynamics of the enlarged quantum system including the apparatus [12–15]. The changes in dynamics induced by the system-apparatus interaction are interpreted as the measurement effects in this formalism. This formalism is referred to as the *dynamical* formalism in this study. The dynamical formalism, which is particularly suitable for analysis of continuous measurements, has also been widely used in the analysis of the QZE and AZE.

Regarding the relations between these two formalisms, it has been revealed that the conventional results are reproducible by the dynamical formalism [12,13], which emphasizes the fact that the projection postulate is not necessarily required for explanation of the Zeno effect. The aim of this study is to bring a transparency to the relation between these two formalisms. In Secs. III and IV, we calculate the measurement-modified decay rates Γ_c and Γ_d based on the conventional and dynamical formalisms, respectively. It is shown that, under the condition that the response time of the apparatus is identical for every decay product (flat response), the dynamical formalism reproduces the conventional results. However, the conventional formalism cannot handle more general measurements, where the response time is not necessarily identical for every decay product; the dynamical formalism is indispensable for analysis of such general measurements.

II. THEORETICAL MODEL

As an example of an unstable quantum state, we employ an excited atom undergoing radiative decay. It should be remarked, however, that the model presented below is applicable to other unstable systems. The Hamiltonian of the system is given, taking $\hbar = c = 1$, by

$$\mathcal{H}_s = \Omega \sigma_+ \sigma_- + \int d\boldsymbol{k}[(g_{\boldsymbol{k}}\sigma_+ b_{\boldsymbol{k}} + \text{H.c.}) + \epsilon_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}}], \qquad (1)$$

where Ω is the atomic transition energy, $\boldsymbol{k}$ denotes the wave vector of an emitted photon, $\epsilon_{\boldsymbol{k}}$ is the energy of a photon in mode $\boldsymbol{k}$, and $g_{\boldsymbol{k}}$ is the atom-photon coupling. Here, the creation operators for atomic excitation and photons are denoted by σ_+ and $b^\dagger_{\boldsymbol{k}}$, respectively. Denoting the vacuum state (no atomic excitation and no photons) by $|0\rangle$, the state vector of the initial unstable state is given by $|i\rangle = \sigma_+|0\rangle$. The form factor of the atom-photon interaction is extracted from Eq. (1) as

$$|g_\mu|^2 = \int d\boldsymbol{k}|g_{\boldsymbol{k}}|^2 \delta(\epsilon_{\boldsymbol{k}} - \mu). \qquad (2)$$

Based on this model, we calculate the measurement-modified decay rate of an excited atom in the conventional manner in Sec. III, and in the dynamical manner in Sec. IV.

*Electronic address: ikuzak@aria.mp.es.osaka-u.ac.jp

III. MEASUREMENT-MODIFIED DECAY RATE BY CONVENTIONAL FORMALISM

In this section, we calculate the measurement-modified decay rate based on the conventional formalism, i.e., by combining the free-decay dynamics determined by $\mathcal{H}_s$ and the projection postulate. In conventional theories on the QZE and AZE, it is often assumed that measurements are repeated periodically with a definite time interval. Here, we treat a more general case, where the measurement interval τ is a stochastic variable with a probability density $P(\tau)$. The probability density is normalized as $\int d\tau\, P(\tau)=1$. The mean interval is hereafter denoted by $\tau_{\rm m}$ $[=\int d\tau\, \tau P(\tau)]$.

When the measurement intervals take the values of $(\tau_1,\tau_2,\ldots,\tau_N)$, the survival probability after the Nth measurement is given by $\Pi_{j=1}^{N} s(\tau_j)$, where $s(t)$ is the survival probability of an excited atom in free evolution, i.e., $s(t) = |\langle i|e^{-i\mathcal{H}_s t}|i\rangle|^2$. Here, we have neglected the revival probability, which is usually extremely small in irreversible processes such as atomic decay. The averaged decay rate is given by

$$\Gamma_{\rm c} = -\left\langle \frac{\ln \prod_{j=1}^{N} s(\tau_j)}{\sum_{j=1}^{N} \tau_j} \right\rangle, \tag{3}$$

where $\langle\cdots\rangle$ means the average over a probability distribution $P(\tau)$. Using the law of large numbers ($\Sigma_{j=1}^{N}\tau_j \simeq N\tau_{\rm m}$ for large N) in the denominator, we can obtain a simplified form of the averaged decay rate:

$$\Gamma_{\rm c}(\tau_{\rm m}) = -\tau_{\rm m}^{-1}\langle \ln s(\tau)\rangle, \tag{4}$$

In the usual discussions of the QZE and AZE, because $\tau_{\rm m}$ is taken to be very small, only the short-time behavior of $s(t)$ is relevant. The short-time survival probability can be evaluated by the perturbation theory as

$$s(t) = 1 - t^2 \int d\mu |g_\mu|^2 \mathrm{sinc}^2[(\mu-\Omega)t/2], \tag{5}$$

when sinc $x = x^{-2}\sin x$. Substituting Eq. (5) into Eq. (4), we can obtain a more transparent form of $\Gamma_{\rm c}(\tau_{\rm m})$:

$$\Gamma_{\rm c}(\tau_{\rm m}) = \int d\mu |g_\mu|^2 f_{\rm c}(\mu), \tag{6}$$

$$f_{\rm c}(\mu) = \tau_{\rm m}^{-1}\langle \tau^2\, \mathrm{sinc}^2[(\mu-\Omega)\tau/2]\rangle, \tag{7}$$

where the decay rate is determined by integrating the form factor with a weight function $f_{\rm c}(\mu)$ [4]. Although the above formula is simple and compact, it is known that the formula covers most theoretical predictions on the Zeno effect, such as the QZE-AZE crossover [5].

Let us see two concrete forms of $f_{\rm c}(\mu)$. First, when $P(\tau)=\delta(\tau-\tau_{\rm m})$, namely, when measurements are repeated periodically, $f_{\rm c}(\mu)$ reduces to the well-known form [4]

$$f_{\rm c1}(\mu) = \tau_{\rm m}\, \mathrm{sinc}^2[(\mu-\Omega)\tau_{\rm m}/2]. \tag{8}$$

Second, we consider a case where τ is a $\gamma(1)$-variate [16] and $P(\tau)$ is given by

$$P(\tau) = \tau_{\rm m}^{-1}\exp(-\tau/\tau_{\rm m}). \tag{9}$$

In this case, $f_{\rm c}(\mu)$ is reduced to a Lorentzian:

$$f_{\rm c2}(\mu) = \frac{2\tau_{\rm m}^{-1}}{|\mu-\Omega+i\tau_{\rm m}^{-1}|^2}. \tag{10}$$

IV. MEASUREMENT-MODIFIED DECAY RATE BY DYNAMICAL FORMALISM

In the previous section, the measurement-modified decay rate is calculated in the conventional formalism. In this section, we discuss the measurement-modified decay rate using the dynamical formalism, taking account of the system-apparatus interaction explicitly. Regarding a concrete type of measurement for atomic decay, we here assume a photodetection measurement. The interaction between photons and the detector is described by the following interaction Hamiltonian:

$$\mathcal{H}_{sd} = \int\int d\boldsymbol{k}\, d\omega[\{(2\pi\tau_{\boldsymbol{k}})^{-1/2} b_{\boldsymbol{k}}^\dagger c_{\boldsymbol{k}\omega} + \text{H.c.}\} + \omega c_{\boldsymbol{k}\omega}^\dagger c_{\boldsymbol{k}\omega}]. \tag{11}$$

where $c_{\boldsymbol{k}\omega}$ is the annihilation operator for the excitation in the detector with the total momentum $\boldsymbol{k}$ and energy ω. In the $\boldsymbol{k}$-conserved form of the photon-detector interaction, it is implicitly assumed that the detector is spatially homogeneous [17]. By this photon-detector interaction, an emitted photon with wave vector $\boldsymbol{k}$ is detected with a response time $\tau_{\boldsymbol{k}}$. A set of response times $\{\tau_{\boldsymbol{k}}\}$ characterizes the performance of the detector.

By analyzing the unitary time evolution determined by the enlarged Hamiltonian $\mathcal{H}_s + \mathcal{H}_{sd}$, one can obtain the measurement-modified decay rate. It has been proved that, due to the photon-detector interaction Hamiltonian $\mathcal{H}_{sd}$, the bare form factor given by Eq. (2) is renormalized to take the following form [14,15]:

$$|\overline{g_\mu}|^2 = \int d\boldsymbol{k} |g_{\boldsymbol{k}}|^2 \frac{(2\pi\tau_{\boldsymbol{k}})^{-1}}{|\mu-\epsilon_{\boldsymbol{k}}-i(2\tau_{\boldsymbol{k}})^{-1}|^2}. \tag{12}$$

The measurement-modified decay rate $\Gamma_{\rm d}$ is given, applying the Fermi golden rule to the renormalized form factor, by

$$\Gamma_d(\{\tau_{\boldsymbol{k}}\}) = 2\pi|\overline{g_\Omega}|^2 = \int d\boldsymbol{k} |g_{\boldsymbol{k}}|^2 \frac{\tau_{\boldsymbol{k}}^{-1}}{|\Omega-\epsilon_{\boldsymbol{k}}-i(2\tau_{\boldsymbol{k}})^{-1}|^2}. \tag{13}$$

This is a general result, which is applicable to any forms of $|g_{\boldsymbol{k}}|^2$, $\epsilon_{\boldsymbol{k}}$, and $\tau_{\boldsymbol{k}}$.

Now we apply the above formula to an idealized situation of *flat* response, where every photon is detected with an identical response time $\tau_{\rm r}$, namely,

$$\tau_{\boldsymbol{k}} = \tau_{\rm r}, \tag{14}$$

regardless of $\boldsymbol{k}$ [12,13]. In this case, the decay rate is recast into the following form:

$$\Gamma_d(\tau_{\rm r}) = \int d\mu |g_\mu|^2 f_d(\mu), \tag{15}$$

$$f_d(\mu) = \frac{\tau_{\mathrm{r}}^{-1}}{|\mu - \Omega - i(2\tau_{\mathrm{r}})^{-1}|^2}, \tag{16}$$

where the decay rate is, again, calculated by integrating the form factor with a weight function $f_{\mathrm{d}}(\mu)$.

V. DISCUSSION

In Sec. III, the measurement-modified decay rate $\Gamma_{\mathrm{c}}(\tau_{\mathrm{m}})$ is calculated based on the conventional formalism, taking account of stochasticity in the measurement intervals. The final form of $\Gamma_{\mathrm{c}}(\tau_{\mathrm{m}})$ is given by Eqs. (6) and (7). On the other hand, in Sec. IV, the measurement-modified decay rate $\Gamma_{\mathrm{d}}(\tau_{\mathrm{r}})$ is calculated based on the dynamical formalism. After imposing the flat-response condition, $\Gamma_{\mathrm{d}}(\tau_{\mathrm{r}})$ is reduced to Eqs. (15) and (16). Now the close connection between the two formalisms is revealed. Both $\Gamma_{\mathrm{c}}(\tau_{\mathrm{m}})$ and $\Gamma_{\mathrm{d}}(\tau_{\mathrm{r}})$ are recast into a unified form, where Γ is given by integrating the original form factor $|g_\mu|^2$ with a weight function $f(\mu)$. Furthermore, the weight functions $f_{\mathrm{c}}(\mu)$ and $f_{\mathrm{d}}(\mu)$ have the following common properties: (i) $f(\mu)$ is a positive function centered at Ω (atomic transition frequency) with a spectral width roughly given by τ_{m}^{-1} or τ_{r}^{-1}, and (ii) $f(\mu)$ is normalized as $\int d\mu\, f(\mu) = 2\pi$. These common properties suggest that the conventional theories on the QZE and AZE can be reproduced by the dynamical formalism, at least qualitatively, provided that the condition of flat response is satisfied. The only difference lies in the functional forms of $f_{\mathrm{c}}(\mu)$ and $f_{\mathrm{d}}(\mu)$, which would result in slight quantitative discrepancy. However, the two results may agree even at a quantitative level in some cases. For example, Eqs. (10) and (16) indicate that a complete agreement is attained by regarding $\tau_{\mathrm{m}} = 2\tau_{\mathrm{r}}$, when $P(\tau)$ is given by Eq. (9).

It should be stressed that equivalence between the two formalisms is guaranteed only under the condition of flat response, Eq. (14). One might wonder why this condition is required. This question is resolved by inspecting the effect of the projection operation on the state vector of the system. By applying the projection postulate, the quantum coherences between the undecayed state $(\sigma_+|0\rangle)$ and the decayed states $(b_{\boldsymbol{k}}^\dagger|0\rangle)$ are lost simultaneously, regardless of $\boldsymbol{k}$. Therefore, the flat response is implicitly assumed in the conventional formalism. For direct measurements, where the unstable system is directly touched by the measurement apparatus, the condition of flat response is satisfied in most cases, so analyses by the conventional formalism are validated. However, the condition of flat response is not necessarily satisfied in general measurement processes, particularly in indirect ones. Therefore, it is expected that the general formula based on the dynamical formalism, Eq. (13), may contain phenomena beyond the conventional wisdom on the QZE and AZE. For example, let us consider a case where the unobserved decay law exactly follows an exponential one $s(t) = e^{-\gamma t}$, which is accomplished when the form factor is a constant function as $|g_\mu|^2 = \gamma/2\pi$. In this case, as is well known, the conventional theory predicts that neither the QZE nor the AZE can be induced. Actually, Eqs. (6) and (7) necessarily predict that the decay rate is unchanged from the unobserved one $(\Gamma_{\mathrm{c}} = \gamma)$, regardless of $P(\tau)$. However, even in this case, it has been demonstrated based on the dynamical formalism that the QZE or AZE may take place when the condition of flat response is not satisfied [14,15], which is a normal situation in real experiments.

[1] B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).
[2] A. M. Lane, Phys. Lett. **99A**, 359 (1983).
[3] B. Kaulakys and V. Gontis, Phys. Rev. A **56**, 1131 (1997).
[4] A. G. Kofman and G. Kurizki, Nature (London) **405**, 546 (2000).
[5] P. Facchi, H. Nakazato, and S. Pascazio, Phys. Rev. Lett. **86**, 2699 (2001).
[6] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A **41**, 2295 (1990).
[7] S. R. Wilkinson, C. F. Bharucha, M. C. Fischer, K. W. Madison, P. R. Morrow, Q. Niu, B. Sundaram, and M. G. Raizen, Nature (London) **387**, 575 (1997).
[8] M. C. Fischer, B. Gutierrez-Medina, and M. G. Raizen, Phys. Rev. Lett. **87**, 040402 (2001).
[9] Y. Aharonov and M. Vardi, Phys. Rev. D **21**, 2235 (1980).
[10] H. Nakazato, T. Takazawa, and K. Yuasa, Phys. Rev. Lett. **90**, 060401 (2003).
[11] N. Erez, Y. Aharonov, B. Reznik, and L. Vaidman, Phys. Rev. A **69**, 062315 (2004).
[12] L. S. Schulman, Phys. Rev. A **57**, 1509 (1998).
[13] P. Facchi and S. Pascazio, Fortschr. Phys. **49**, 941 (2001).
[14] K. Koshino and A. Shimizu, Phys. Rev. Lett. **92**, 030401 (2004).
[15] K. Koshino, Phys. Rev. Lett. **93**, 030401 (2004).
[16] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge University Press, Cambridge, U.K., 1995).
[17] B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992).